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Summability of connected correlation functions of coupled lattice fields

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Abstract

We consider two nonindependent random fields ψ and ϕ defined on a countable set Z . For instance, $Z = \mathbb{Z}^d$ or $Z = \mathbb{Z}^d \times I$, where I denotes a finite set of possible “internal degrees of freedom” such as spin. We prove that, if the cumulants of ψ and ϕ enjoy a certain decay property, then all joint cumulants between ψ and ϕ are ℓ_2 -summable in the precise sense described in the text. The decay assumption for the cumulants of ψ and ϕ is a restricted ℓ_1 summability condition called ℓ_1 -clustering property. One immediate application of the results is given by a stochastic process $\psi_t(x)$ whose state is ℓ_1 -clustering at any time t : then the above estimates can be applied with $\psi = \psi_t$ and $\phi = \psi_0$ and we obtain uniform in t estimates for the summability of time-correlations of the field. The above clustering assumption is obviously satisfied by any ℓ_1 -clustering stationary state of the process, and our original motivation for the control of the summability of time-correlations comes from a quest for a rigorous control of the Green-Kubo correlation function in such a system. A key role in the proof is played by the properties of non-Gaussian Wick polynomials and their connection to cumulants.

1 Introduction and physical motivation

In many problems of physical interest, the basic dynamic variable is a random field. In addition to proper stochastic processes, such as particles evolving according to Brownian motion, the random field could describe for instance a density of particles of a Hamiltonian system with random initial data or after time-averaging.

One particular instance of the second kind is the *Green-Kubo formula* which connects the transport coefficients, such as thermal conductivity, to integrals over *equilibrium time-correlations* of the current observable of the relevant conserved quantity, for instance, of the energy current. The equilibrium time-correlations are moments of current fields between time zero and some later time. The current fields are generated by distributing the initial data according to some fixed equilibrium measure and then solving the evolution equations: this yields a random field, even when the time-evolution itself is deterministic.

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Each correlation function, evaluated as an expectation of a product of field values at the given points, has also a corresponding *connected correlation function* which is defined as the cumulant of the random variables given by these field values. In statistical mechanics, the connected correlations functions are often called *Ursell functions*. For an extensive discussion about the topic, see for instance [1], [2].

Hence, the control of correlation functions of random fields is a central problem for a rigorous study of transport properties. One approach which has been used both in practical applications and in direct mathematical studies, is given by Boltzmann transport equations. It is usually derived from the microscopic system by using moment hierarchies, such as the BBGKY hierarchy, and then ignoring higher order moments to close the hierarchy of evolution equations. Although apparently quite a powerful method, it has not been possible to give any meaningful general estimates for the accuracy or for regions of applicability of such closure approximations.

The present work arose as part of a project aiming at a rigorous derivation of a Boltzmann transport equation in the kinetic scaling limit of the weakly nonlinear discrete Schrödinger equation (DNLS). This system describes the evolution of a complex lattice field $\psi_t(\mathbf{x})$, with $\mathbf{x} \in \mathbb{Z}^d$ and $t \geq 0$, by requiring that it satisfies the Hamiltonian evolution equations

$$i\partial_t\psi_t(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{Z}^d} \alpha(\mathbf{x} - \mathbf{y})\psi_t(\mathbf{y}) + \lambda|\psi_t(\mathbf{x})|^2\psi_t(\mathbf{x}), \quad (1.1)$$

where the function α determines the “hopping amplitudes” and $\lambda > 0$ is a coupling constant. A kinetic scaling limit with a suitably chosen closure assumption predicts that the Green-Kubo correlation function of the energy density of this system satisfies a linearized phonon Boltzmann equation in the limit; the explicit form of the Boltzmann collision operator and discussion about the approximations involved is given in Sections 5 and 6 in [3], and we refer to [4, 5] for more details about the linearization procedure.

The method used in the derivation of transport properties in such weak coupling limits are, naturally, based on perturbation expansions. Advances have recently been made in controlling the related oscillatory integrals (see for instance [6, 7, 8]), but for nonlinear evolution equations of the present type a major obstacle has been the lack of useful *a priori* bounds for the connected correlation functions.

For instance, Schwarz inequality estimates of moments in the “remainder terms” of finitely expanded moment hierarchies have been used for this purpose for time-stationary initial data in [7]. The technique was inspired by the bounds from unitarity of the time-evolution of certain linear evolution equations first employed for the random Schrödinger equation in [6] and later extended to other similar models such as the Anderson model [9] and a classical harmonic lattice with random mass perturbations [10]. However, as argued in [3], using moments instead of cumulants to develop the hierarchy could lead to loss of an important decay property which is valid for cumulants but not for moments; we shall discuss this point further in Section 2.

Moreover, recently *a priori* bounds have turned out to be crucial also in the framework of systems of interacting hard spheres: for example, in [11, 12] the authors employed *a priori* estimates for marginal distributions and for related functions which, as the cumulants, quantify the decay of correlations in order to study the hydrodynamics of such models. A similar strategy has been followed by Pulvirenti and Simonella in [13] where they use bounds on some suitable centered moments to study the correlation errors in the Boltzmann–Grad limit. More details about the kinetic theory and scaling limits of hard sphere systems can be found in the monograph by Spohn [14].

In the present contribution we derive a generic result which allows to bound joint correlations of two random fields in terms of estimates involving only the decay properties of each of the fields separately. These estimates are immediately applicable for obtaining *uniform in time* *a priori* bounds for time-correlation functions of time-stationary fields. In particular, they imply that if the initial state of the field is distributed according to an equilibrium measure which is ℓ_1 -clustering, then all time-correlations are ℓ_2 -summable. The precise assumptions are described in Section 2 and the result in Theorem 2.1 there.

If both fields are Gaussian and translation invariant, more direct estimates involving discrete Fourier-transform become available. We use this in Section 3 to give an explicit example which shows that ℓ_1 -clustering of the fields does not always extend to their joint correlations, hence showing that the increase of the power from ℓ_1 -clustering to ℓ_2 -summability of the joint correlations in Theorem 2.1 below is in general necessary.

The result is a corollary of a bound which proves summability of cumulants of any observable with finite variance with ℓ_p -clustering fields, for $p = 1$ and $p = 2$. The $p = 2$ case is more involved than the $p = 1$ case, since the present bound requires taking the sum in a weighted ℓ_2 -space. The precise statements and all proofs are given in Section 4.

Initial measures which are ℓ_2 -clustering arise naturally whenever Fourier transforms are involved in their definition, such as in the Gaussian translation invariant case described in Section 3 below. However, physical equilibrium measures—in particular in their high temperature phases—often have correlation functions which are even exponentially decaying in the separation (i.e., *exponentially clustering*) and thus ℓ_1 -clustering; see Ref. [15] for an example. Moreover, it is proven in [16] that the canonical Gibbs measures of the DNLS equation are ℓ_1 -clustering for a certain range of hopping amplitudes and equilibrium parameters. Hence, in the study of equilibrium time-correlations, the easier ℓ_1 -clustering often suffices.

The proof of Theorem 2.1 is based on the representation of cumulants via Wick polynomials. We rely on the results and notations of [3], and for convenience of the reader we have summarized the relevant items in Appendix A. We also discuss possible applications of these bounds in Section 5.

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2 Notations and mathematical setting

We consider here *complex lattice fields* $\psi : Z \rightarrow \mathbb{C}$ where Z is any nonempty *countable* index set. We focus on this particular setup since it is the one most directly relevant for physical applications: common examples would be $Z = \mathbb{Z}^d$ and $Z = \mathbb{Z}^d \times I$, where I denotes a finite set of possible “internal degrees of freedom” such as spin. The setup can also cover more abstract index sets, such as the sequence of coefficients in the Karhunen–Loève decomposition of a stochastic process [17, 18, 19], or distribution-valued random fields evaluated at suitably chosen sequence of test-functions (details about the definition and properties of general random fields can be found for instance in [20, 21] and in other sources discussing the Bochner–Minlos theorem).

We also assume that the field is *closed under complex conjugation*: to every $x \in Z$ there is some $x_* \in Z$ for which $\psi(x)^* = \psi(x_*)$. If needed, this can always be achieved by replacing the original index set Z by $Z \times \{-1, 1\}$ and defining a new field Ψ by setting $\Psi(x, 1) = \psi(x)$ and $\Psi(x, -1) = \psi(x)^*$. This procedure was in fact used in [3, 7] to study the DNLS example mentioned above, resulting in the choice $Z = \mathbb{Z}^d \times \{-1, 1\}$.

A *random lattice field* on Z is then a collection of random variables $\psi(x)$, $x \in Z$, on the probability space $(\Omega, \mathcal{M}, \mu)$, where Ω denotes the sample space, \mathcal{M} the σ -algebra of measurable events, and μ the probability measure. We consider here two random fields ψ and ϕ which are

defined on the same probability space, and the measure μ then defines a *coupling* between the fields ψ and ϕ .¹ We denote the expectation over the measure μ by \mathbb{E} .

The n :th *connected correlation function* u_n of the field ψ is a map $u_n : Z^n \rightarrow \mathbb{C}$ which is defined as the cumulant of the n random variables obtained by evaluating the field at the argument points; explicitly,

$$u_n(x) := \kappa[\psi(x_1), \dots, \psi(x_n)], \quad x \in Z^n. \quad (2.1)$$

We employ here the notations and basic results for cumulants and the related Wick polynomials, as derived in [3]: a summary of these is also included in Appendix A.

In physics, one often encounters random fields defined on the d -dimensional cubic lattice, with $Z = \mathbb{Z}^d$. One could then study the decay properties of such functions as $|x| \rightarrow \infty$ by using the standard ℓ_p -norms over $(\mathbb{Z}^d)^n$. However, this is typically too restrictive for physical applications: it would imply in particular that both the first and the second cumulant, i.e., the mean and the variance, of the random variable $\psi(x)$ decay as $|x| \rightarrow \infty$, and thus the field would be almost surely “asymptotically zero” at infinity. Instead, many stationary measures arising from physical systems are *spatially translation invariant*: the expectation values remain invariant if all of the fields $\psi(x)$ are replaced by $\psi(x + x_0)$ for any given $x_0 \in \mathbb{Z}^d$. Since this implies also translation invariance of all correlation functions, they cannot decay at infinity then, unless the field is almost surely zero *everywhere*.

To cover also such nondecaying stationary states, one uses instead of the direct ℓ_p -norms of the function u_n , the so-called ℓ_p -*clustering norms* of the field ψ defined as follows: for $1 \leq p < \infty$ and $n \in \mathbb{N}_+$ we set

$$\|\psi\|_p^{(n)} := \sup_{x_0 \in Z} \left[\sum_{x \in Z^{n-1}} |\kappa[\psi(x_0), \psi(x_1), \dots, \psi(x_{n-1})]|^p \right]^{1/p}, \quad (2.2)$$

and define analogously $\|\psi\|_p^{(1)} := \sup_{x_0 \in Z} |\mathbb{E}[\psi(x_0)]|$ (note that $\kappa[X] = \mathbb{E}[X]$ for any random variable X). We shall also use the corresponding $p = \infty$ norms, which coincide with the standard sup-norms of u_n , namely, $\|\psi\|_\infty^{(n)} = \sup_{x \in Z^n} |\kappa[\psi(x_1), \psi(x_2), \dots, \psi(x_n)]| = \|u_n\|_\infty$. Since the norms concern L^p -spaces over a counting measure, they are decreasing in p , i.e., $\|\psi\|_p^{(n)} \geq \|\psi\|_{p'}^{(n)}$ if $p \leq p'$. (This follows from the bound $|u_n(x)| \leq \|\psi\|_p^{(n)}$, valid for all x and p .)

For a translation invariant measure with $Z = \mathbb{Z}^d$, we can translate x_0 to the origin in the definition (2.2), and, by a change of variables $x_n = x_0 + y_n$, obtain the simpler expression

$$\|\psi\|_p^{(n)} = \left[\sum_{y \in (\mathbb{Z}^d)^{n-1}} |\kappa[\psi(0), \psi(y_1), \dots, \psi(y_{n-1})]|^p \right]^{1/p}. \quad (2.3)$$

The summation here goes over the *displacements* y_i of the argument x_i from the reference position $x_0 = 0$. The definition is tailored for random fields which become asymptotically independent for far apart regions of the lattice, i.e., when $|y_i| \rightarrow \infty$ above. For translation non-invariant measures, finiteness of the norm (2.2) yields a uniform estimate for the speed of asymptotic independence of the field. Let us use the opportunity to stress that it is crucial to use the cumulants, not moments, above: similar moments of the field would not decay as the separation grows, even if the field values would become independent (see [3] for more discussion about this point).

We now call a random field ψ ℓ_p -*clustering* if $\|\psi\|_p^{(n)} < \infty$ for all $n = 1, 2, \dots$. In particular, this requires that all of the cumulants, which define the connected correlation functions u_n , need to exist. From the iterative definition of cumulants mentioned in the Appendix, or from the inversion

¹To follow the mathematical terminology more precisely, we first need to define measures ν_ψ and ν_ϕ by setting ν_ψ to be the marginal distribution of μ obtained by considering only expectations of functions of ψ (i.e., after “integrating over other degrees of freedom”), yielding a probability measure on \mathbb{C}^Z , and defining ν_ϕ analogously using the field ϕ . Then ν_ψ determines the distribution of the field ψ , ν_ϕ the distribution of the field ϕ , and μ defines a coupling between ν_ψ and ν_ϕ in the standard mathematical sense.

formula expressing cumulants in terms of moments, it clearly suffices that $\mathbb{E}[|\psi(x)|^n] < \infty$ for all $x \in Z$. We also say that the field ψ is ℓ_p -clustering up to order m if $\|\psi\|_p^{(n)} < \infty$ for all $n \leq m$. For such a field, we use the following constants to measure its “magnitude”: we set

$$M_N(\psi; p) := \max_{1 \leq n \leq N} \left(\frac{1}{n!} \|\psi\|_p^{(n)} \right)^{1/n}. \quad (2.4)$$

Clearly, the definition yields an increasing sequence in N up to the same order in which the field is ℓ_p -clustering. We use the constants M_N to control the increase of the clustering norms. It is conceivable that in special cases other choices beside (2.4) could be used with the estimates below to arrive at sharper bounds than those stated in the theorems. However, the above choice is convenient for our purposes since it leads to simple combinatorial estimates, increasing typically only factorially in the degree of the cumulant. It is possible to think of the numbers M_N as measuring the range of values the field can attain. For instance, if $Z = \{0\}$ and $\psi(0)$ is a random variable which almost surely belongs to the interval $[-R, R]$ with $R > 0$, then $M_n(\psi; p)$ is independent of p (since there is only one point 0) and $M_n = c_n R$ where c_n remains order one, uniformly in n (see, for instance, Lemma C.1 in [10]).

After these preliminaries, we are ready to state the main result:

Theorem 2.1 *Suppose ψ and ϕ are random lattice fields which are closed under complex conjugation and defined on the same probability space. Assume that ϕ is ℓ_1 -clustering and ψ is ℓ_∞ -clustering, both up to order $2N$ for some $N \in \mathbb{N}_+$. Then their joint cumulants satisfy the following ℓ_2 -estimate for any $n, m \in \mathbb{N}_+$ for which $n, m \leq N$,*

$$\sup_{x' \in Z^m} \left[\sum_{x \in Z^n} |\kappa[\psi(x'_1), \dots, \psi(x'_m), \phi(x_1), \dots, \phi(x_n)]|^2 \right]^{1/2} \leq (\mathfrak{M}_{m,n} \gamma^m)^{n+m} (n+m)!, \quad (2.5)$$

where $\mathfrak{M}_{m,n} := \max(M_{2m}(\psi; \infty), M_{2n}(\phi; 1))$ and $\gamma = 2e \approx 5.44$. In particular, all of the above sums are then finite.

Loosely speaking, one can say that an ℓ_1 -clustering random field can have at worst ℓ_2 -summable joint correlations. We have stated the result in a form which assumes that the field ψ is ℓ_∞ -clustering. As mentioned above, the clustering norms are decreasing in the index: hence, the above result also holds if ψ is ℓ_q -clustering for any $1 \leq q < \infty$. One could then also replace the constants $\mathfrak{M}_{m,n}$ using the corresponding ℓ_q -clustering norms, $\max(M_{2m}(\psi; q), M_{2n}(\phi; 1))$. However, these constants are always larger than $\mathfrak{M}_{m,n}$ and thus can only worsen the bound.

This result is a consequence of a more general covariance bound given in Theorem 4.1. There we also give a version of the estimate for fields ϕ which are merely ℓ_2 -clustering. The price to pay for the relaxation of the norms is an appearance of a weight factor in the ℓ_2 -summation, see Theorem 4.3 for the precise statement. Before going into the details of the proofs, let us go through a special case clarifying the assumptions and the result.

3 An example: translation invariant Gaussian lattice fields

In this section, we consider real valued Gaussian random fields ψ and ϕ on $Z = \mathbb{Z}$ and assume that both fields have a zero mean and are invariant under spatial translations. Their joint measure is then determined by giving three functions $F_1, F_2, G \in \ell_2(\mathbb{Z}, \mathbb{R})$ for which

$$\langle \psi(x)\psi(y) \rangle = F_1(x-y), \quad \langle \phi(x)\phi(y) \rangle = F_2(x-y), \quad \langle \psi(x)\phi(y) \rangle = G(x-y). \quad (3.1)$$

The covariance operator needs to be positive semi-definite. By first using Parseval’s theorem and then computing the eigenvalues of the remaining 2×2 -matrix, we find that this is guaranteed by requiring that the Fourier-transforms of the above functions, all of which belong to $L^2(\mathbb{T})$, satisfy almost everywhere

$$\widehat{F}_1(k) \geq 0, \quad \widehat{F}_2(k) \geq 0, \quad |\widehat{G}(k)|^2 \leq \widehat{F}_1(k)\widehat{F}_2(k). \quad (3.2)$$

These three conditions hence suffice for the existence of a unique Gaussian measure on functions on \mathbb{Z} satisfying (3.1); details about such constructions are given for instance in [21, 22, 23].

The last condition restricts the magnitude of the correlations, and it implies that if each of the above fields is ℓ_2 -clustering, then their correlations are ℓ_2 -summable (simply because then $\widehat{G}(k) \in L^2(\mathbb{T})$, and thus its inverse Fourier transform gives a function $G \in \ell_2(\mathbb{Z})$). Hence, one might wonder if the main theorem could, in fact, be strengthened to show that ℓ_1 -clustering of the fields implies ℓ_1 -summability of the joint correlations. The following example shows that this is not the case.

3.1 ℓ_1 -clustering fields whose joint correlations are not ℓ_1 -summable

Let us consider two i.i.d. Gaussian fields ψ and ϕ whose correlations are determined by the function

$$G(x) = \frac{1}{\pi x} \sin\left(\frac{\pi}{2}x\right), \quad x \neq 0, \quad G(0) = \frac{1}{2}. \quad (3.3)$$

For such i.i.d. fields $F_1(x) = \mathbb{1}(x = 0) = F_2(x)$ which is equivalent to $\widehat{F}_1(k) = 1 = \widehat{F}_2(k)$ for all $k \in \mathbb{T}$. Now for all $x \in \mathbb{Z}$, clearly

$$G(x) = \int_{-1/4}^{1/4} dk e^{i2\pi x k}, \quad (3.4)$$

and thus $\widehat{G}(k) = \mathbb{1}(|k| < \frac{1}{4}) \leq 1 = \sqrt{\widehat{F}_1(k)\widehat{F}_2(k)}$. Therefore, such G indeed defines a possible correlation between the fields ψ and ϕ .

For such Gaussian fields, all cumulants of order different from $n = 2$ are zero. We also have $\sup_{x \in \mathbb{Z}} \sum_{y \in \mathbb{Z}} |F_1(x - y)| = 1$, and, as $F_2 = F_1$, both fields are ℓ_1 -clustering, with $\|\psi\|_1^{(2)} = 1 = \|\phi\|_1^{(2)}$ and $\|\psi\|_1^{(n)} = 0 = \|\phi\|_1^{(n)}$ for any other n . However, their joint correlations satisfy for any $x' \in \mathbb{Z}$

$$\sum_{x \in \mathbb{Z}} |\kappa[\psi(x'), \phi(x)]| = \sum_{y \in \mathbb{Z}} |G(y)| = \frac{1}{2} + 2 \sum_{y=1}^{\infty} \frac{1}{\pi y} \left| \sin\left(\frac{\pi}{2}y\right) \right| = \frac{1}{2} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} = \infty. \quad (3.5)$$

Thus the joint correlations are not ℓ_1 -summable.

In contrast, $\sup_{x'} \sum_x |\kappa[\psi(x'), \phi(x)]|^2 < \infty$ since it is equal to $\sum_y |G(y)|^2$ and $G \in \ell^2(\mathbb{Z})$.

4 ℓ_2 -summability of joint correlations of ℓ_p -clustering fields

Theorem 4.1 *Consider a random lattice field ϕ on a countable set Z , defined on a probability space $(\Omega, \mathcal{M}, \mu)$ and closed under complex conjugation. Suppose that ϕ is ℓ_p -clustering up to order $2N$ for some $N \in \mathbb{N}_+$, and let $M_N(\phi; p)$ be defined as in (2.4). Suppose also $X \in L^2(\mu)$, i.e., X is a random variable with finite variance.*

1. *If $p = 1$ and $n \leq N$, we have a bound*

$$\left[\sum_{x \in Z^n} |\kappa[X, \phi(x_1), \dots, \phi(x_n)]|^2 \right]^{1/2} \leq \sqrt{\text{Cov}(X^*, X)} M_{2n}(\phi; 1) e^n \sqrt{(2n)!}. \quad (4.1)$$

2. *If $p = 2$ and $n \leq N$, we have a bound*

$$\sup_{x' \in Z^n} \left[\sum_{x \in Z^n} |\Phi_n(x', x)| |\kappa[X, \phi(x_1), \dots, \phi(x_n)]|^2 \right]^{1/2} \leq \sqrt{\text{Cov}(X^*, X)} M_{2n}(\phi; 2) e^{2n} (2n)!, \quad (4.2)$$

where $\Phi_n(x', x) := \mathbb{E}[\phi(x'_1)^* \phi(x'_2)^* \cdots \phi(x'_n)^* : \phi(x_1) \phi(x_2) \cdots \phi(x_n)]$.

The key argument in the proof uses Wick polynomial representation of the above cumulants. Namely, a direct consequence of the truncated moments-to-cumulants formula given in Proposition A.1 in the Appendix, is that

$$\kappa[X, \phi(x_1), \dots, \phi(x_n)] = \mathbb{E}[X : \phi(x_1)\phi(x_2) \cdots \phi(x_n) :] = \mathbb{E}[:X : \phi(x_1)\phi(x_2) \cdots \phi(x_n) :]. \quad (4.3)$$

We have relied here on $n \geq 1$, which guarantees that $\mathbb{E}[:\phi(x_1)\phi(x_2) \cdots \phi(x_n) :] = 0$, and hence we could center the random variable X without cost, i.e., replace it by $:X := X - \mathbb{E}[X]$. The Proposition can be applied here since now $\mathbb{E}[|X| \prod_{i=1}^n |\phi(x_i)|] < \infty$ by the Schwarz inequality estimate $\mathbb{E}[|X| \prod_{i=1}^n |\phi(x_i)|]^2 \leq \mathbb{E}[|X|^2] \mathbb{E}[\prod_{i=1}^n |\phi(x_i)|^2]$ where the first factor is finite since $X \in L^2(\mu)$, and the second factor is finite since ϕ is assumed to be ℓ_p -clustering up to order $2n$.

Applying Schwarz inequality in (4.3) yields a bound

$$|\kappa[X, \phi(x_1), \dots, \phi(x_n)]|^2 \leq \mathbb{E}[:X :]^2 \mathbb{E}[:\phi(x_1)\phi(x_2) \cdots \phi(x_n) :]^2 = \text{Cov}(X^*, X) \Phi_n(x, x). \quad (4.4)$$

Hence, the theorem is obviously true if $\Phi_n(x, x)$ decreases sufficiently rapidly with “increasing” x . However, this is typically too restrictive: since $\Phi_1(x, x) = \mathbb{E}[:\phi(x) :]^2 = \text{Var}(\phi(x))$, this would require that the field ϕ becomes asymptotically deterministic. The proof below combines suitably chosen test functions with the above Schwarz estimate and results in bounds which only require summability of $\Phi_n(x', x)$ in x for a fixed x' . Such summability is guaranteed by the ℓ_p -clustering of the field, and the rest of the proof consists of controlling the combinatorial factors which relate these two concepts together, cf. Lemma 4.2.

Let us stress that the above result is typically *not* true if *moments* are used there instead of cumulants. The above Schwarz inequality estimates would be straightforward for moments; in fact, such a Schwarz estimate was a key method in [7] to separate time-evolved fields from their time-zero counterparts in products of these fields. However, the functions resulting from such Schwarz estimates are of the type $\mathbb{E}[\prod_{i=1}^n |\phi(x_i)|^2]$ and for these to be summable in x the field not only has to become asymptotically deterministic, but it has to even vanish. Cumulants of ℓ_p -clustering fields would, on the other hand, be summable, but there is no obvious way of generalizing the Schwarz inequality bounds for cumulants. The missing ingredient is here provided by the Wick polynomial representation (4.3).

Proof: There is a natural Hilbert space structure associated with correlations of the present type. We begin with test-functions $f : Z^n \rightarrow \mathbb{C}$ which have a *finite* support, and define for them a (semi-)norm by the formula

$$\begin{aligned} \|f\|_{\phi, n}^2 &:= \mathbb{E} \left[\left| \sum_{x \in Z^n} f(x) : \phi(x)^{J_n} : \right|^2 \right] = \sum_{x', x \in Z^n} f(x')^* f(x) \Phi_n(x', x), \\ \Phi_n(x', x) &:= \mathbb{E} \left[: \phi^*(x')^{J'_n} : : \phi(x)^{J_n} : \right], \end{aligned} \quad (4.5)$$

where $J_n = \{1, 2, \dots, n\} = J'_n$, and thus we have $\phi(x)^{J_n} := \phi(x_1)\phi(x_2) \cdots \phi(x_n)$, $\phi^*(x')^{J'_n} := \phi(x'_1)^* \phi(x'_2)^* \cdots \phi(x'_n)^*$. For the definition, we do not yet need any summability properties of the field ϕ , it suffices that all the expectations in $\Phi_n(x', x)$ are well-defined for all x', x . By the truncated moment-to-cumulants expansion of Wick polynomials, as given in Proposition A.1 in the Appendix, we have here

$$\Phi_n(x', x) = \sum_{\pi \in \mathcal{P}(J'_n + J_n)} \prod_{S \in \pi} (\kappa[\phi^*(x')_{A'}, \phi(x)_A] \mathbb{1}(A' \neq \emptyset, A \neq \emptyset))_{A'=S|J'_n, A=S|J_n}, \quad (4.6)$$

where the notation $S|J_n$ refers to the subsequence composed out of the indices belonging to J_n in the cluster S of the partition π of $J'_n + J_n$.² The additional restrictions $A', A \neq \emptyset$ in the

²If one has distinct labels in J'_n and J_n , achievable always by relabelling of one of the sets, one can safely take here $J'_n + J_n = J'_n \cup J_n$, $\mathcal{P}(J'_n + J_n)$ equal to the ordinary partitions of the set $J'_n \cup J_n$, and also $S|J_n = S \cap J_n$. Here, we have opted to use the definitions from [3] instead of explicit relabellings.

product arise from the fact that if either of them is violated, then the corresponding cluster S is contained entirely in either J'_n or J_n , and vice versa. The partitions containing such a cluster are precisely those which are missing from the moments to cumulants formula by the Wick polynomial construction. Therefore, Φ_n is finite, as soon as all cumulants up to order $2n$ are finite. On the other hand, this is already guaranteed by the assumed ℓ_p -clustering of the field ϕ . For notational simplicity, let us drop the name of the field ϕ from the norm $\|f\|_{\phi,n}$.

The norm can be associated with a scalar product using the polarization identity, and we can then use it to define a Hilbert space \mathcal{H}_n by completion and dividing out the functions with zero norm, if the above formula gives only a semi-norm. The elements of \mathcal{H}_n are thus functions $f : Z^n \rightarrow \mathbb{C}$ with $\|f\| < \infty$ (or their equivalence classes in the semi-norm case when every f and g with $\|f - g\| = 0$ needs to be identified). However, since we do not use these Hilbert spaces directly, let us skip the details of the construction.

We begin with joint correlations of the type $G(x) := \mathbb{E}[Y : \phi(x)^{J_n} :]$ where $Y \in L^2(\mu)$ is a random variable. Here $G(x)$ is well defined due to the Schwarz inequality estimate $\mathbb{E}[|Y| : \phi(x)^{J_n} :]^2 \leq \mathbb{E}[|Y|^2] \Phi_n(x, x)$. If $f : Z^n \rightarrow \mathbb{C}$ has a finite support, we define

$$\Lambda[f] := \sum_{x \in Z^n} G(x) f(x) = \mathbb{E} \left[Y \sum_{x \in Z^n} : \phi(x)^{J_n} : f(x) \right]. \quad (4.7)$$

Applying the Schwarz inequality as above yields an upper bound

$$|\Lambda[f]|^2 \leq \mathbb{E}[|Y|^2] \mathbb{E} \left[\left| \sum_{x \in Z^n} f(x) : \phi(x)^{J_n} : \right|^2 \right] = \mathbb{E}[|Y|^2] \|f\|_n^2. \quad (4.8)$$

Here, by the definition of the norm, we obtain an unweighted ℓ_2 -estimate by using Hölder's inequality as follows

$$\begin{aligned} \|f\|_n^2 &\leq \sum_{x', x \in Z^n} |f(x')| |f(x)| |\Phi_n(x', x)| \leq \sqrt{\sum_{x', x \in Z^n} |f(x')|^2 |\Phi_n(x', x)|} \sqrt{\sum_{x', x \in Z^n} |f(x)|^2 |\Phi_n(x', x)|} \\ &\leq \sum_{x \in Z^n} |f(x)|^2 \sup_{x' \in Z^n} \sum_{x \in Z^n} |\Phi_n(x', x)|, \end{aligned} \quad (4.9)$$

where we have used the obvious symmetry property $\Phi_n(x', x)^* = \Phi_n(x, x')$. As shown below, in Lemma 4.2, ℓ_1 -clustering of the field ϕ in fact implies that there is $c_n < \infty$ such that $\sup_{x' \in Z^n} \sum_{x \in Z^n} |\Phi_n(x', x)| \leq c_n$ (the explicit dependence of c_n on the clustering norms is given in the Lemma). Hence, we can conclude that $|\Lambda[f]| \leq \sqrt{c_n \mathbb{E}[|Y|^2]} \|f\|_{\ell_2}$. Thus, thanks to the Riesz representation theorem, Λ can be extended into a unique functional belonging to the dual of the Hilbert space $\ell_2(Z^n)$, and hence there is a vector $\Psi \in \ell_2(Z^n)$ such that $\Lambda[f] = \sum_{x \in Z^n} \Psi(x)^* f(x)$ and $\|\Psi\|_{\ell_2} \leq \sqrt{c_n \mathbb{E}[|Y|^2]}$. Then necessarily $G(x) = \Psi(x)^*$ for all x , and thus $G \in \ell_2(Z^n)$ as well, with a bound

$$\sqrt{\sum_{x \in Z^n} |G(x)|^2} \leq \sqrt{c_n \mathbb{E}[|Y|^2]}. \quad (4.10)$$

If $Y = :X:$, we have $G(x) = \mathbb{E}[:X : \phi(x)^{J_n} :] = \kappa[X, \phi(x_1), \dots, \phi(x_n)]$ as explained in (4.3), and also $\mathbb{E}[|Y|^2] = \mathbb{E}[:X^* : :X:] = \kappa[X^*, X] = \text{Cov}(X^*, X)$. Hence, (4.10) implies the bound stated in the first item.

For the weighted result, we apply (4.8) for specially constructed test functions f . Let F be any finite subset of Z^n and choose an arbitrary point $y \in Z^n$. Then $f(x) = \mathbb{1}(x \in F) |\Phi_n(y, x)| G(x)^*$ has finite support and

$$\Lambda[f] = \sum_{x \in F} |G(x)|^2 |\Phi_n(y, x)| \leq \sqrt{\mathbb{E}[|Y|^2]} \|f\|_n < \infty. \quad (4.11)$$

On the other hand, we obtain the following estimate for $\|f\|_n$

$$\begin{aligned}
\|f\|_n^2 &= \sum_{x', x \in F} G(x)^* G(x') |\Phi_n(y, x')| |\Phi_n(y, x)| |\Phi_n(x', x)| \\
&\leq \sqrt{\sum_{x', x \in F} |G(x)|^2 |\Phi_n(y, x)| |\Phi_n(y, x')| |\Phi_n(x', x)|} \sqrt{\sum_{x', x \in F} |G(x')|^2 |\Phi_n(y, x')| |\Phi_n(y, x)| |\Phi_n(x', x)|} \\
&\leq \sum_{x \in F} |G(x)|^2 |\Phi_n(y, x)| \left(\sup_{x' \in Z^n} \sqrt{\sum_{x \in F} |\Phi_n(x', x)|^2} \right)^2, \tag{4.12}
\end{aligned}$$

where we have used $\Phi_n(x', x)^* = \Phi_n(x, x')$ and the Schwarz inequality in the last estimate.

As shown below, in Lemma 4.2, ℓ_2 -clustering of the field ϕ implies $\sqrt{\sum_{x \in Z^n} |\Phi_n(x', x)|^2} \leq c'_n < \infty$ where the explicit dependence of c'_n on the clustering norms is given in the Lemma. Therefore, $\Lambda[f] \leq c'_n \sqrt{\mathbb{E}[|Y|^2]} \sqrt{\Lambda[f]}$. Since $0 \leq \Lambda[f] < \infty$ for any subset F , we can conclude that the estimate $\sqrt{\Lambda[f]} \leq c'_n \sqrt{\mathbb{E}[|Y|^2]}$ also holds. Thus by using subsets $F = F_R$, which are constructed by choosing the first R elements from a fixed enumeration of Z^n , and then taking $R \rightarrow \infty$, we obtain that

$$\sqrt{\sum_{x \in Z^n} |G(x)|^2 |\Phi_n(y, x)|} \leq c'_n \sqrt{\mathbb{E}[|Y|^2]} < \infty, \tag{4.13}$$

for all $y \in Z^n$. This implies the statement in the second item. \square

Lemma 4.2 *Suppose that the field ϕ is closed under complex conjugation and ℓ_p -clustering up to order $2n$, for some $p \in [1, \infty]$ and $n \geq 1$. Then, for any $x' \in Z^n$,*

$$\|\Phi_n(x', \cdot)\|_{\ell_p} \leq \sum_{\pi \in \mathcal{P}(J_{2n})} \prod_{S \in \pi} \|\phi\|_p^{(|S|)} \leq M_{2n}(\phi; p)^{2n} e^{2n} (2n)!, \tag{4.14}$$

where $J_{2n} = \{1, 2, \dots, 2n\}$ and $\mathcal{P}(J_{2n})$ denotes the collection of its partitions.

Proof: Let us consider some fixed $x' \in Z^n$. We apply the Minkowski inequality to (4.6), as a function of x , and conclude that

$$\|\Phi_n(x', \cdot)\|_{\ell_p} \leq \sum_{\pi \in \mathcal{P}(J'_n + J_n)} \|F(x', \cdot; \pi)\|_{\ell_p} \tag{4.15}$$

where

$$F(x', x; \pi) := \prod_{S \in \pi} (|\kappa[\phi^*(x')_{A'}, \phi(x)_A]| \mathbb{1}(A' \neq \emptyset, A \neq \emptyset))_{A'=S|J'_n, A=S|J_n}. \tag{4.16}$$

Let us first consider the case $p < \infty$. For any $\pi \in \mathcal{P}(J'_n + J_n)$ yielding a nonzero F , the restrictions of its clusters with J_n , $A = S|J_n$ in the above formula, form a partition of J_n . Let us denote this partition by π_2 . Hence, we can use this partition to reorder the summation over $x \in Z^n$ into iterative summation over $x_A \in Z^A$ for $A \in \pi_2$. Applied to (4.16) this yields

$$\sum_{x \in Z^n} |F(x', x; \pi)|^p = \prod_{S \in \pi} \left(\mathbb{1}(A' \neq \emptyset, A \neq \emptyset) \sum_{x_A \in Z^A} |\kappa[\phi^*(x')_{A'}, \phi(x)_A]|^p \right)_{A'=S|J'_n, A=S|J_n}. \tag{4.17}$$

Since the field ϕ is closed under complex conjugation, for each $S \in \pi$ the sum over x_A is equal to $\sum_{x \in Z^A} |\kappa[\phi(y')_{J_{|A'|}}, \phi(x)_A]|^p$ where $y' := ((x'_i)_*)_{i \in A'}$ defines an explicit relabelling. As $A' \neq \emptyset$,

we may choose an element $j \in A'$. We then denote $x_0 = (x'_j)_*$ and estimate the sum with an ℓ_p -clustering norm as follows

$$\sum_{x \in Z^A} |\kappa[\phi^*(x')_{A'}, \phi(x)_A]|^p \leq \sum_{y \in Z^{|A'|+|A|-1}} \sum_{x \in Z^A} \left| \kappa[\phi(x_0), \phi(y)_{J_{|A'|+1}}, \phi(x)_A] \right|^p \leq \left(\|\phi\|_p^{(|A|+|A'|)} \right)^p. \quad (4.18)$$

Since $|A'| + |A| = |S|$, we can conclude that, if $p < \infty$,

$$\|F(x', \cdot; \pi)\|_{\ell_p} \leq \prod_{S \in \pi} \|\phi\|_p^{(|S|)}. \quad (4.19)$$

The corresponding estimate for $p = \infty$ is a straightforward consequence of $|\kappa[\phi^*(x')_{A'}, \phi(x)_A]| \leq \|\phi\|_\infty^{(|A'|+|A|)}$ which was discussed in Section 2 after Eq. (2.2).

Therefore, we can now conclude that the first inequality in (4.14) holds. By the definition in (2.4), we can then apply an upper bound

$$\|\phi\|_p^{(m)} \leq m! M_m(\phi; p)^m \leq m! M_{2n}(\phi; p)^m,$$

for any $m \leq 2n$. If $\pi \in \mathcal{P}(J'_n + J_n)$, we have $|S| \leq 2n$ for any $S \in \pi$, and thus

$$\prod_{S \in \pi} \|\phi\|_p^{(|S|)} \leq M_{2n}(\phi; p)^{\sum_{S \in \pi} |S|} \prod_{S \in \pi} |S|! = M_{2n}(\phi; p)^{2n} \prod_{S \in \pi} |S|!. \quad (4.20)$$

A combinatorial estimate shows that

$$\sum_{\pi \in \mathcal{P}(J_{2n})} \prod_{S \in \pi} |S|! \leq (2n)! e^{2n} \quad (4.21)$$

(a proof of the inequality is available for instance in the proof of Lemma 7.3 in [7]). Therefore, we have proven also the second inequality in (4.14), concluding the proof of the Lemma. \square

The following theorem contains the already stated Theorem 2.1 in the item 1. The remarks after the Theorem at the end of Section 2 hold also in this case. In particular, it is obviously valid for any ℓ_q -clustering field ψ , as long as $1 \leq q \leq \infty$.

Theorem 4.3 *Consider two random lattice fields $\phi(x)$ and $\psi(x)$, $x \in Z$ for a countable Z , defined on the same probability space $(\Omega, \mathcal{M}, \mu)$ and each closed under complex conjugation. Suppose that ϕ is ℓ_p -clustering and ψ is ℓ_∞ -clustering up to order $2N$ for some $N \in \mathbb{N}_+$. Let M_N be defined as in (2.4). Then their joint cumulants satisfy the following ℓ_2 -estimates for any $n, m \in \mathbb{N}_+$ for which $n, m \leq N$:*

1. If $p = 1$, we have a bound

$$\sup_{x' \in Z^m} \left[\sum_{x \in Z^n} |\kappa[\psi(x'_1), \dots, \psi(x'_m), \phi(x_1), \dots, \phi(x_n)]|^2 \right]^{1/2} \leq (\mathfrak{M}_{m,n} \gamma^m)^{n+m} (n+m)! \quad (4.22)$$

where $\mathfrak{M}_{m,n} := \max(M_{2m}(\psi; \infty), M_{2n}(\phi; 1))$ and $\gamma = 2e$.

2. If $p = 2$, we have a bound

$$\begin{aligned} \sup_{x' \in Z^m, y \in Z^n} \left[\sum_{x \in Z^n} |\Phi_n(y, x)| |\kappa[\psi(x'_1), \dots, \psi(x'_m), \phi(x_1), \dots, \phi(x_n)]|^2 \right]^{1/2} \\ \leq (\mathfrak{M}_{m,n} \gamma^m)^{2(n+m)} ((n+m)!)^2 \end{aligned} \quad (4.23)$$

where $\Phi_n(y, x) := \mathbb{E}[\phi(y_1)^* \phi(y_2)^* \dots \phi(y_n)^* : \phi(x_1) \phi(x_2) \dots \phi(x_n)]$, and we set $\mathfrak{M}_{m,n} := \max(M_{2m}(\psi; \infty), M_{2n}(\phi; 2))$ and $\gamma = 2e$.

Proof: We will proceed by induction over m . Let us recall the above definition of Φ_n and define analogously $\Psi_m(y', x') := \mathbb{E}[\psi(y'_1)^* \psi(y'_2)^* \cdots \psi(y'_m)^* : \psi(x'_1) \psi(x'_2) \cdots \psi(x'_m)]$. In particular, then we can apply Theorem 4.1 with $X = \psi(x'_1)$. By Lemma 4.2, then $\mathbb{E}[|X|^2] = \Psi_1(x'_1, x'_1) \leq M_2(\psi; \infty)^2 e^2 2!$, and thus, say for $\gamma = 2e$, both items 1 and 2 can be seen to hold for $m = 1$ and any $n \leq N$ thanks to Theorem 4.1 and the estimate $(2n)! \leq ((2n)!)^2 = 2^{2n}(n!)^2$.

As an induction hypothesis, we consider some $1 < m \leq N$ and assume that the thesis holds for values up to $m - 1$ with any $n \leq N$. We also give the details only for the first ℓ_1 -clustering case, i.e., with $p = 1$.

Let us decompose the cumulant using Proposition A.1. Namely, consider

$$\mathcal{P}(x', x) := \mathbb{E}[\psi(x'_1) \cdots \psi(x'_m) : \phi(x_1) \cdots \phi(x_n)],$$

for which

$$\kappa[\psi(x'_1), \dots, \psi(x'_m), \phi(x_1), \dots, \phi(x_n)] = \mathcal{P}(x', x) - \mathcal{Q}(x', x),$$

with

$$\mathcal{Q}(x', x) := \sum_{\pi \in \mathcal{P}(J'_m + J_n)} \mathbb{1}(|\pi| > 1) \prod_{S \in \pi} (\kappa[\psi(x')_{A'}, \phi(x)_A] \mathbb{1}(A' \neq \emptyset, A \neq \emptyset))_{A'=S|J'_m, A=S|J_n}.$$

Then we can conclude from the Minkowski inequality that

$$\left[\sum_{x \in Z^n} |\kappa[\psi(x'_1), \dots, \psi(x'_m), \phi(x_1), \dots, \phi(x_n)]|^2 \right]^{1/2} \leq \|\mathcal{P}(x', \cdot)\|_{\ell_2} + \|\mathcal{Q}(x', \cdot)\|_{\ell_2}. \quad (4.24)$$

We now estimate $\|\mathcal{P}(x', \cdot)\|_{\ell_2}$ using item 1 in Theorem 4.1 with $X = \psi(x'_1) \cdots \psi(x'_m)$. Clearly, then

$$\mathcal{P}(x, x') = \mathbb{E}[X : \phi(x_1) \cdots \phi(x_n)] = \kappa[X, \phi(x_1), \dots, \phi(x_n)], \quad (4.25)$$

and, by applying first Theorem 4.1 and then Lemma 4.2,

$$\|\mathcal{P}(x', \cdot)\|_{\ell_2} \leq \sqrt{\Psi_m(x', x')} M_{2n}(\phi; 1)^n e^n \sqrt{(2n)!} \leq M_{2m}(\psi; \infty)^m M_{2n}(\phi; 1)^n e^{n+m} \sqrt{(2m)!(2n)!}. \quad (4.26)$$

Note that $(2n)! \leq 2^{2n}(n!)^2$ and $n!m! \leq (n+m)!$, so, recalling the definition of $\mathfrak{M}_{m,n}$, we have

$$\|\mathcal{P}(x', \cdot)\|_{\ell_2} \leq (\mathfrak{M}_{m,n} 2e)^{n+m} (n+m)!. \quad (4.27)$$

To control the second term in (4.24), we first use the Minkowski inequality to the sum over the partitions, as in the proof of Theorem 4.1, yielding

$$\begin{aligned} & \|\mathcal{Q}(x', \cdot)\|_{\ell_2} \\ & \leq \sum_{\pi \in \mathcal{P}(J'_m + J_n)} \mathbb{1}(|\pi| > 1) \prod_{S \in \pi} \left(\left[\sum_{x \in Z^A} |\kappa[\psi(x')_{A'}, \phi(x)_A]|^2 \right]^{1/2} \mathbb{1}(A' \neq \emptyset, A \neq \emptyset) \right)_{A'=S|J'_m, A=S|J_n}. \end{aligned}$$

In the final expression all the cumulants $\kappa[\psi(x')_{A'}, \phi(x)_A]$ are such that $|A'| < m$ and $|A| < n$. Therefore, the induction hypothesis can be applied to estimate their ℓ_2 -norms:

$$\left[\sum_{x \in Z^A} |\kappa[\psi(x')_{A'}, \phi(x)_A]|^2 \right]^{1/2} \leq (\mathfrak{M}_{m',n'} \gamma^{m'})^{|S|} |S|!_{m'=|S|J'_m, n'=|S|J_n}.$$

Note that $\mathfrak{M}_{m,n}$ is non-decreasing in m , and $m' \leq m-1$; hence,

$$\begin{aligned} \|\mathcal{Q}(x', \cdot)\|_{\ell_2} &\leq \sum_{\pi \in \mathcal{P}(J'_m + J_n)} \mathbb{1}(|\pi| > 1) \prod_{S \in \pi} (\mathfrak{M}_{m',n'} \gamma^{m'})^{|S|} |S|! |_{m'=|S|J'_m, n'=|S|J_n} \\ &\leq \sum_{\pi \in \mathcal{P}(J'_m + J_n)} \prod_{S \in \pi} (\mathfrak{M}_{m,n} \gamma^{m-1})^{|S|} |S|! = (\mathfrak{M}_{m,n} \gamma^{m-1})^{m+n} \sum_{\pi \in \mathcal{P}(J'_m + J_n)} \prod_{S \in \pi} |S|! \\ &\leq (\mathfrak{M}_{m,n} \gamma^{m-1})^{m+n} e^{n+m} (n+m)! = (\mathfrak{M}_{m,n} \gamma^m)^{m+n} (e/\gamma)^{n+m} (n+m)! \end{aligned}$$

where in the last inequality we have used (4.21). Collecting the two estimates together we have proven

$$\|\mathcal{P}(x', \cdot)\|_{\ell_2} + \|\mathcal{Q}(x', \cdot)\|_{\ell_2} \leq \mathfrak{M}_{m,n}^{n+m} (n+m)! [(2e)^{n+m} + (e/\gamma)^{n+m} \gamma^{m(m+n)}]. \quad (4.28)$$

In order to close the induction, we need to choose γ such that

$$(2e)^{n+m} + (e/\gamma)^{n+m} \gamma^{m(m+n)} \leq \gamma^{m(m+n)}.$$

Since $m \geq 2$, it suffices to set, for instance, $\gamma = 2e$.

The proof of the second item, with $p = 2$, is essentially the same: one only needs to replace the flat ℓ_2 -norm by the above weighted ℓ_2 -norm containing the factor $|\Phi_n(y, x)|$ (which can also be understood as integrals over the corresponding weighted counting measure over Z), and to apply item 2 in Theorem 4.1 instead of item 1 there. To reach the same combinatorial estimates, we can reduce the resulting second powers of $|S|!$ to sums over first powers via the bound $\prod_{S \in \pi} |S|! \leq (n+m)!$ which is valid for any partition $\pi \in \mathcal{P}(J'_m + J_n)$. \square

5 Discussion

Suppose that for each $t \geq 0$ there is given $\psi_t(x)$ which is a random field over a countable set Z . Suppose also that all ψ_t are identically distributed, according to an ℓ_1 -clustering measure; such fields arise, for instance, from stochastic processes by choosing the initial data from a stationary measure which is ℓ_1 -clustering. For such a system Theorem 2.1 implies that any cumulant of the form

$$\kappa[\psi_0(x'_1), \dots, \psi_0(x'_m), \psi_t(x_1), \dots, \psi_t(x_n)] \quad (5.1)$$

belongs to $\ell_2(Z^n)$ in x and its norm is uniformly bounded in t by a constant which depends only on the initial measure.

To have a concrete example, let us return to the discrete NLS evolution and its equilibrium time-correlations, as discussed in the Introduction. At the time of writing, we are not aware of a rigorous definition of the infinite volume dynamics for an equilibrium measure of the DNLS. However, DNLS evolution is well-defined on any finite periodic lattice, and there is a range of hopping amplitudes and equilibrium parameters for which the corresponding thermal Gibbs states are ℓ_1 -clustering, uniformly in the lattice size, as proven in [16]. Therefore, it seems likely that there are harmonic couplings for which also the DNLS evolution equations on \mathbb{Z}^d with initial data distributed according to a stationary measure can be solved almost surely. In addition, it should be possible to define the stationary measure so that it is ℓ_1 -clustering and translation invariant.

Define $\psi_t(\mathbf{x}, 1) := \psi(\mathbf{x})$ and $\psi_t(\mathbf{x}, -1) := \psi_t(\mathbf{x})^*$ where $\mathbf{x} \in \mathbb{Z}^d$ and $\psi(\mathbf{x})$ evolves in time according to (1.1). Let the initial measure be a stationary measure which is assumed to be ℓ_1 -clustering and translation invariant. Setting $\sigma = \pm 1$, $X = \psi_0(0, \sigma_0)$ and $Z = \mathbb{Z}^d \times I$ with $I = \{1, -1\}$, Theorem 4.1 implies that the $\ell_2((\mathbb{Z}^d \times I)^{n-1})$ -norm of any time-correlation function of the form

$$F_{t,n,\sigma_0}(x, \sigma) := \kappa[\psi_0(0, \sigma_0), \psi_t(\mathbf{x}_1, \sigma_1), \dots, \psi_t(\mathbf{x}_{n-1}, \sigma_{n-1})], \quad x \in (\mathbb{Z}^d)^{n-1} \quad (5.2)$$

is uniformly bounded in time and the bound depends only on the initial measure.

For any such ℓ_1 -clustering stationary measure, we can then study the evolution of the functions F_{t,n,σ_0} using the above results. Referring to [3] for details, for instance $f_t(\mathbf{x}) := \kappa[\psi_0(0, -1), \psi_t(\mathbf{x}, 1)] = \mathbb{E}[\psi_0(0, -1): \psi_t(\mathbf{x}, 1)]$ would then satisfy an evolution equation

$$i\partial_t f_t(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{Z}^d} \alpha(\mathbf{x} - \mathbf{y}) f_t(\mathbf{y}) + \lambda g_t(\mathbf{x}), \quad (5.3)$$

where $g_t(\mathbf{x}) := \mathbb{E}[\psi_0(0, -1): \psi_t(\mathbf{x}, -1)\psi_t(\mathbf{x}, 1)\psi_t(\mathbf{x}, 1)]$. Applying Proposition A.1, g_t can be represented in terms of the constant $\mathbb{E}[\psi_t(\mathbf{x}, 1)]$ and the functions $F_{t,n,-1}$, with $n = 2, 3, 4$. Using the above estimates, it then follows that there is a constant C such that $\|g_t\|_{\ell_2} \leq C$ for all t . Taking a Fourier-transform of (5.3) and solving it in Duhamel form implies that $f_t = \mathcal{F}f_t$ satisfies

$$\widehat{f}_t(\mathbf{k}) = e^{-it\widehat{\alpha}(\mathbf{k})} \widehat{f}_0(\mathbf{k}) - i\lambda \int_0^t ds e^{-i(t-s)\widehat{\alpha}(\mathbf{k})} \widehat{g}_s(\mathbf{k}), \quad \mathbf{k} \in \mathbb{T}^d. \quad (5.4)$$

For stable harmonic interactions one needs to have $\widehat{\alpha}(\mathbf{k}) \geq 0$. Therefore, for such systems we may conclude, without any complicated analysis of oscillatory integrals or graph expansion of cumulants, that the harmonic evolution dominates the behavior of f_t up to times of order λ^{-1} . More precisely, we find that the ℓ_2 -norm of the error is bounded by

$$\left\| f_t - \mathcal{F}^{-1}(e^{-it\widehat{\alpha}} \widehat{f}_0) \right\|_{\ell_2} = \left\| \widehat{f}_t - e^{-it\widehat{\alpha}} \widehat{f}_0 \right\|_{L^2(\mathbb{T}^d)} \leq Ct\lambda, \quad (5.5)$$

for all $t \geq 0$.

The above example perhaps does not appear very significant: after all, it is simply stating that the nonlinearity acts as a perturbation in ℓ_2 -norm with its “natural” strength, having an effect of order λt to the time-evolution. Let us however stress that without the present *a priori* bounds there seems to be no other alternative to prove this than to resort to the heavy machinery of time-dependent perturbation theory with Feynman graph classification of oscillatory integrals and careful applications of moment hierarchies, see [7] for a detailed example for DNLS.

Another important property hidden in the bound (5.5) is the fact that $\widehat{f}_t(\mathbf{k})$ is a *function*, and not a distribution. This would *not* be true in general if instead of cumulants we would have used moments to define f_t ; if $\mathbb{E}[\psi_0(0)] \neq 0$ and the initial state is translation invariant, already at $t = 0$ the Fourier transform of $\mathbb{E}[\psi_0(0, -1)\psi_t(\mathbf{x}, 1)]$ is a distribution proportional to the Dirac delta $\delta(\mathbf{k})$. The fact that the cumulants produce functions, which are uniformly bounded in ℓ_2 , allows not only taking Fourier transforms but also simplifying the study of nonlinear terms in the hierarchies as products of distributions are notoriously difficult to control rigorously.

In conclusion, it would be of interest to study more carefully the above combinatorial bounds. We do not claim that the above constants, or their dependence on the orders n and m , should be optimal, and there could be room for significant improvement there, possibly of importance in problems requiring the full infinite order cumulant hierarchy. Also, it is not clear what are the optimal powers and weights for the summability of the correlations as the clustering power p of the field ϕ is varied. These questions could prove to be hard to resolve in the greatest generality, but we remain optimistic that already the present bounds suffice to control the time-evolution of cumulants in some of the above mentioned open transport problems.

A Cumulants and Wick polynomials

We have collected in this Appendix the main results relating to Wick polynomials which were used in the text. The goal is not to give an exhaustive exposition of the topic, but rather to list the minimal amount of definitions and properties needed for the proofs here. In the Gaussian case, one can also identify the Wick polynomials as arising from an orthogonalization procedure, with applications in Wiener chaos expansions and Malliavin calculus [22]. The present general, non-Gaussian case, is not directly connected to orthogonal polynomials, and leads to more complicated

combinatorial expansions: we refer to [24] for a recent review of these from the point of view of probability theory and for more results how graphical representations may be used to facilitate the analysis involving Wick polynomials.

Let us consider a collection y_j , $j \in J$ where J is some fixed nonempty index set, of real or complex random variables on some probability space $(\Omega, \mathcal{M}, \mu)$. Then for any sequence of indices, $I = (i_1, i_2, \dots, i_n) \in J^n$, we use the following shorthand notations to label monomials of the above random variables:

$$y^I = y_{i_1} y_{i_2} \cdots y_{i_n} = \prod_{k=1}^n y_{i_k}, \quad y^\emptyset := 1 \quad \text{if } I = \emptyset. \quad (\text{A.1})$$

As already explained in an earlier footnote, we consider sequences of indices and not sets of indices in order to avoid more cumbersome notations involving relabelling of elements whenever an element is repeated in a sequence. We will however continue to use set-like notations for subsequences and partitions of sequences. To be precise, these notations are valid only after one has added a unique label for each element of the sequence. For this, we introduce a collection \mathcal{J} which consists of those finite subsets $A \subset \mathbb{N} \times J$ with the property that if $(n, j), (n', j') \in A$ and $(n, j) \neq (n', j')$ then $n \neq n'$. The empty sequence is identified with $\emptyset \in \mathcal{J}$. For nonempty sets, the natural number in the first component serves as a distinct label for each member in A and their order determines the order of the elements in the sequence.

For any $I \in \mathcal{J}$ we denote the corresponding moment by $\mathbb{E}[y^I]$, and the related cumulant by

$$\kappa[y_I] = \kappa_\mu[y_I] = \kappa[y_{i_1}, y_{i_2}, \dots, y_{i_n}]. \quad (\text{A.2})$$

The corresponding Wick polynomial is denoted by

$$:y^I: = :y^I:_\mu = :y_{i_1} y_{i_2} \cdots y_{i_n}:. \quad (\text{A.3})$$

Both $\kappa[y^I]$ and $:y^I:$ can be defined recursively if $I \in \mathcal{J}$ is such that $\mathbb{E}[|y^E|] < \infty$ for all $E \subset I$ (see [3]). Explicitly, it suffices to require that

$$:y^I: = y^I - \sum_{E \subsetneq I} \mathbb{E}[y^{I \setminus E}] :y^E:, \quad (\text{A.4})$$

and, choosing some $x \in I$,

$$\kappa[y_I] = \mathbb{E}[y^I] - \sum_{E: x \in E \subsetneq I} \mathbb{E}[y^{I \setminus E}] \kappa[y_E]. \quad (\text{A.5})$$

Let us also recall that both cumulants and Wick polynomials are multilinear and permutation invariant.

If the random variables y_j , $j = 1, 2, \dots, n$, have joint exponential moments, then moments, cumulants and Wick polynomials can also be easily generated by differentiation of their respective generating functions which are

$$G_m(\lambda) := \mathbb{E}[e^{\lambda \cdot x}], \quad g_c(\lambda) := \ln G_m(\lambda) \quad \text{and} \quad G_w(\lambda; y) := \frac{e^{\lambda \cdot y}}{\mathbb{E}[e^{\lambda \cdot x}]} = e^{\lambda \cdot y - g_c(\lambda)}. \quad (\text{A.6})$$

By evaluation of the I -th derivative at zero, we have

$$\mathbb{E}[y^I] = \partial_\lambda^I G_m(0), \quad \kappa[y_I] = \partial_\lambda^I g_c(0) \quad \text{and} \quad :y^I: = \partial_\lambda^I G_w(0; y), \quad (\text{A.7})$$

where “ ∂_λ^I ” is a shorthand notation for $\partial_{\lambda_{i_1}} \partial_{\lambda_{i_2}} \cdots \partial_{\lambda_{i_n}}$.

It is remarkable that expectations of products of Wick polynomials can be expanded in terms of cumulants, merely cancelling some terms from the standard moments-to-cumulants expansion. The following result, proven as Proposition 3.8 in [3], details the result using the above notations:

Proposition A.1 Assume that the measure μ has all moments of order N , i.e., suppose that $\mathbb{E}[|y^I|] < \infty$ for all $I \in \mathcal{I}$ with $|I| \leq N$. Suppose $L \geq 1$ is given and consider a collection of $L+1$ index sequences $J', J_\ell \in \mathcal{I}$, $\ell = 1, \dots, L$, such that $|J'| + \sum_\ell |J_\ell| \leq N$. Then for $I := \sum_{\ell=1}^L J_\ell + J'$ (with the implicit identification of J_ℓ and J' with the set of its labels in I) we have

$$\mathbb{E} \left[\prod_{\ell=1}^L :y^{J_\ell}: y^{J'} \right] = \sum_{\pi \in \mathcal{P}(I)} \prod_{A \in \pi} (\kappa[y_A] \mathbb{1}(A \not\subset J_\ell \ \forall \ell)) . \quad (\text{A.8})$$

In words, the constraint determined by the characteristic functions on the right hand side of (A.8) amounts to removing from the standard cumulant expansion all terms which have any clusters internal to one of the sets J_ℓ . For instance, thanks to Proposition A.1, if we consider the expectation of the product of two second order Wick polynomials, we get $\mathbb{E}[:y_1 y_2: :y_3 y_4:] = \kappa(y_1, y_3) \kappa(y_2, y_4) + \kappa(y_1, y_4) \kappa(y_2, y_3) + \kappa(y_1, y_2, y_3, y_4)$.

Proposition A.1 turns out to be a powerful technical tool, used several times in the proofs of Theorems 4.1 and 4.3.

References

- [1] D. Ruelle, *Statistical mechanics: Rigorous results*, W. A. Benjamin, Inc., New York- Amsterdam, 1969.
- [2] B. Simon, *The statistical mechanics of lattice gases. Vol. I*, Princeton Series in Physics, Princeton University Press, Princeton, NJ, 1993.
- [3] J. Lukkarinen and M. Marozzi, *Wick polynomials and time-evolution of cumulants*, J. Math. Phys. **57** (2016) 083301.
- [4] H. Spohn, *The phonon Boltzmann equation, properties and link to weakly anharmonic lattice dynamics*, J. Stat. Phys. **124**(2–4) (2006) 1041–1104.
- [5] J. Lukkarinen, *Kinetic theory of phonons in weakly anharmonic particle chains*, in “Thermal transport in low dimensions: from statistical physics to nanoscale heat transfer”, Lecture Notes in Physics vol. 921 (2016) 159–214, Physics volume (S. Lepri ed.).
- [6] L. Erdős and H.-T. Yau, *Linear Boltzmann equation as the weak coupling limit of a random Schrödinger equation*, Commun. Pure Appl. Math. **53**(6) (2000) 667–735.
- [7] J. Lukkarinen and H. Spohn, *Weakly nonlinear Schrödinger equation with random initial data*, Invent. Math. **183**(1) (2011) 79–188.
- [8] L. Erdős, M. Salmhofer, and H.-T. Yau, *Quantum diffusion of the random Schrödinger evolution in the scaling limit I. The non-recollision diagrams*, Acta Math. **200**(2) (2008) 211–277.
- [9] T. Chen, *Localization lengths and Boltzmann limit for the Anderson model at small disorders in dimension 3*, J. Stat. Phys. **120** (2005) 279–337.
- [10] J. Lukkarinen and H. Spohn, *Kinetic limit for wave propagation in a random medium*, Arch. Ration. Mech. Anal. **183**(1) (2007) 93–162.
- [11] T. Bodineau, I. Gallagher and L. Saint-Raymond, *The Brownian motion as the limit of a deterministic system of hard-spheres*, Invent. Math. **203**(2) (2016) 493–553.
- [12] T. Bodineau, I. Gallagher and L. Saint-Raymond, *From hard sphere dynamics to the Stokes-Fourier equations: an L^2 analysis of the Boltzmann-Grad limit*, arXiv:1511.03057, preprint (2015).

- [13] M. Pulvirenti and S. Simonella, *The Boltzmann–Grad limit of a hard sphere system: analysis of the correlation error*, Invent. Math. **207**(3) (2017) 1135–1237.
- [14] H. Spohn, *Large Scale Dynamics of Interacting Particles*. Springer, Berlin, 1991.
- [15] J. L. Lebowitz and O. Penrose, *Analytic and clustering properties of thermodynamic functions and distribution functions for classical lattice and continuum systems*, Comm. Math. Phys. **11**(2) (1968) 99–124.
- [16] A. Abdesselam, A. Procacci, and B. Scoppola, *Clustering bounds on n -point correlations for unbounded spin systems*, J. Stat. Phys. **136**(3) (2009) 405–452.
- [17] P. Lévy, *Processus Stochastiques et Mouvement Brownien. Suivi d’une note de M. Loève*. Gauthier-Villars, Paris, 1948.
- [18] M. Loève, *Probability theory. II*. Springer-Verlag, New York-Heidelberg, fourth edition, 1978. Graduate Texts in Mathematics, Vol. 46.
- [19] R. J. Adler and J. E. Taylor, *Random Fields and Geometry*. Springer-Verlag New York, 2007.
- [20] J. Glimm and A. Jaffe, *Quantum Physics: A Functional Integral Point of View*. Springer, New York, second edition, 1987.
- [21] V. I. Bogachev, *Gaussian Measures*. American Mathematical Society, 1998.
- [22] S. Janson, *Gaussian Hilbert Spaces*. Cambridge University Press, 1997.
- [23] P. Abrahamsen, *A review of Gaussian random fields and correlation functions*. Technical Report 917, Norwegian Computing Center, Oslo, 1997.
- [24] G. Peccati and M. S. Taqqu, *Wiener Chaos: Moments, Cumulants and Diagrams: A survey with Computer Implementation*. Bocconi & Springer Series. Springer, 2011.